Distributed QR decomposition framework for training Support Vector Machines

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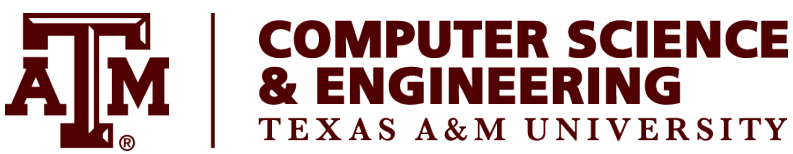

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Introduction

Distributed large-scale QP based Optimization Problems

Some applications of Quadratic Programming (QP) are

1. Least Square approximations
2. Regression Analysis
3. Portfolio Optimization
4. Support Vector Machines
5. Optimal Control

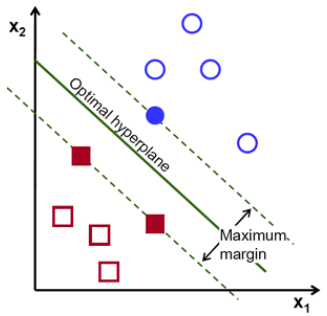
Support Vector Machines (SVM)

We focus on distributed data analytics using SVM

- Supervised machine learning model (data+label)
- Widely used for data classification for its high efficiency
- Popular for multivariate non-linear datasets (kernel SVM)
- Have been extended for tasks like regression analysis (SVR), principal component analysis etc.

SVM as a QP problem

- SVM is a convex optimization problem (QP)
- Solves for maximal separating hyperplane as a classifier



- Maps training vectors into a high dimensional space via a nonlinear function (kernel SVM)
- Hence, solving for *dual* (rather than *primal*) form is preferred using "kernel trick"

Specifically, we focus on the

two-class soft margin SVM with l_2-regularization and l_2-loss

SVM formulation

training dataset, $\mathcal{D} = \{(x_i, y_i), i = 1, ..., n\}$
input data matrix, $X = \{x_i \in \mathbb{R}^d, i = 1...n\}$, d-dimensional space
class label vector, $y = \{y_i \in \{-1, 1\}, i = 1...n\}$

dual SVM

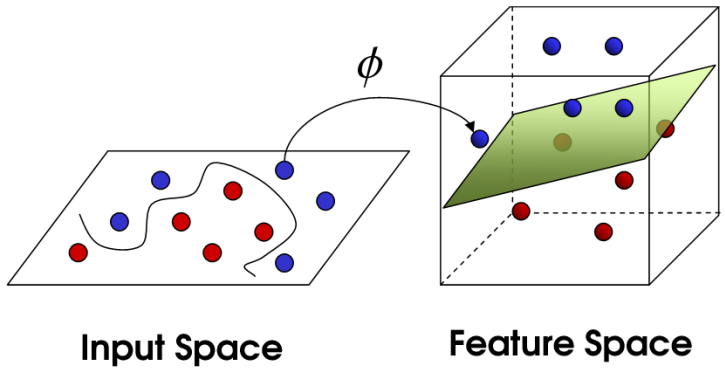
$$\min_{\alpha} \frac{1}{2}\alpha^T \Big(diag(y) \times \mathbf{K} \times diag(y)^T\Big)\alpha + \frac{1}{2}\alpha^T \Big(\frac{1}{2C} I_n\Big)\alpha + e^T\alpha \quad (1)$$

$$\text{subject to} \qquad -I_n\alpha \leq \mathbf{0}_n$$

where, α is a vector of *dual* variables
$e = -\mathbf{1}_n$
$C > 0$ is penalty parameter for misclassification
$\mathbf{K} = \{k(x_i, x_j), \forall i, j = 1...n\}$ is positive definite matrix (mostly)
$k()$ **represents the Mercer kernel function - linear/non-linear**

Kernel SVM for non-linear data

Kernel function:
$k(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle$
where, $\phi()$ is a mapping generally not known or inefficient to compute.



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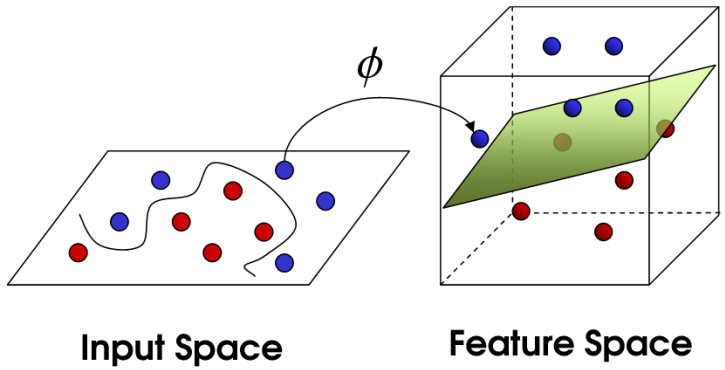

However,
$k(x_i, x_j)$ is known and easier to compute ("Kernel trick").

- Linear kernel : $k(x_i, x_j) = \langle x_i, x_j \rangle$
- Radial Basis Function (RBF) kernel:
  $k(x_i, x_j) = \exp(-\gamma \|x_i - x_j\|^2)$, where, γ is hyperparameter

Measures "similarity" between two data points in the Feature space

Challenges

For large sample size n, Kernel methods become unfeasible because

1. K requires $O(n^2)$ memory and
2. it incurs computational cost of $O(n^3)$ to solve such problems

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2. it incurs computational cost of $O(n^3)$ to solve such problems

Go for **Low Rank Kernel Approximation !**

Low $p-$rank approximation of K

$$K \approx AA^T, \text{ where, } A \in \mathbb{R}^{n \times p} \text{ and } p \ll n.$$

We use MEKA [Si, 2014] for memory efficient and lower approximation error compared to Nyström methods etc.

Recall,

dual SVM

$$\min_{\alpha} \frac{1}{2}\alpha^T \Big(diag(y) \times \mathbf{K} \times diag(y)^T \Big)\alpha + \frac{1}{2}\alpha^T \Big(\frac{1}{2C} I_n \Big)\alpha + e^T \alpha$$

$$\text{subject to} \quad -I_n \alpha \leq \mathbf{0}_n$$

Recall,

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$$\min_{\alpha} \frac{1}{2}\alpha^T \left(diag(y) \times \mathbf{K} \times diag(y)^T \right)\alpha + \frac{1}{2}\alpha^T \left(\frac{1}{2C} I_n \right)\alpha + e^T\alpha$$

$$\text{subject to} \quad -I_n\alpha \leq \mathbf{0}_n$$

Substitute, $K \approx AA^T$ and define, $\hat{A} = diag(y) \times A$

approximated dual SVM

$$\min_{\alpha} \frac{1}{2}\alpha^T \left(\hat{A}\hat{A}^T \right)\alpha + \frac{1}{2}\alpha^T \left(\frac{1}{2C} I_n \right)\alpha + e^T\alpha \tag{2}$$

$$\text{subject to} \quad -I_n\alpha \leq \mathbf{0}_n$$

Motivation

Goal

To devise a fast and memory-efficient distributed framework to train large-scale SVM

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To devise a fast and memory-efficient distributed framework to train large-scale SVM

Our Contribution

1. QRSVM: QR decomposition framework for **memory-efficient** modeling and training of SVM
2. Optimal step size calculation for **fast convergence** of Dual Ascent method which iteratively solves the SVM problem
3. Distributed QRSVM: designing distributed QR decomposition and parallel Dual Ascent techniques for **distributed SVM training**
4. Compared training time of distributed QRSVM with competing distributed methods; PSVM and P-packSVM

QRSVM

Memory-efficient modeling and training of SVM

$\hat{A} \in \mathbb{R}^{n \times p}$ with $p \ll n$ has a tall and skinny (TS) structure

QR decomposition

$$\hat{A} = QR,$$

where, $Q \in \mathbb{R}^{n \times n}$ is Orthogonal matrix

$R \in \mathbb{R}^{n \times p}$ is Upper Triangular matrix

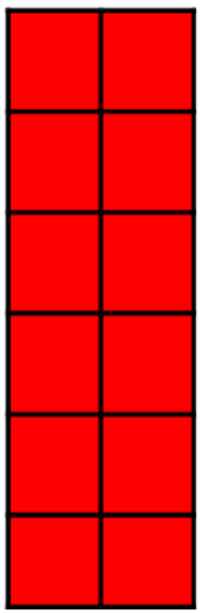


Figure: $\hat{A}$

Q, $O(n^2) \rightarrow p$-Householder reflector vectors, $O(np)$

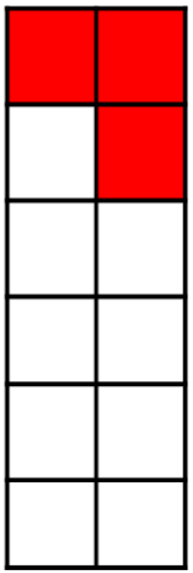


Figure: R

Formulation

Recall,

approximated *dual* SVM

$$\min_{\alpha} \frac{1}{2}\alpha^T\left(\hat{A}\hat{A}^T\right)\alpha + \frac{1}{2}\alpha^T\left(\frac{1}{2C}I_n\right)\alpha + e^T\alpha$$
$$\text{subject to} \quad -I_n\alpha \leq \mathbf{0}_n$$

Now, Substitute $\hat{A} = QR$

Formulation

Substituting $\hat{A} = QR$

$$\min_{\alpha} \frac{1}{2}\alpha^T \left(QRR^T Q^T\right)\alpha + \frac{1}{2}\alpha^T \left(\frac{1}{2C} I_n\right)\alpha + e^T \alpha$$

$$\text{subject to} \quad -I_n \alpha \leq \mathbf{0}_n$$

Define, $\hat{\alpha} = Q^T \alpha$, $\hat{e} = Q^T e$ and using $Q^T Q = I_n$

Formulation

QRSVM

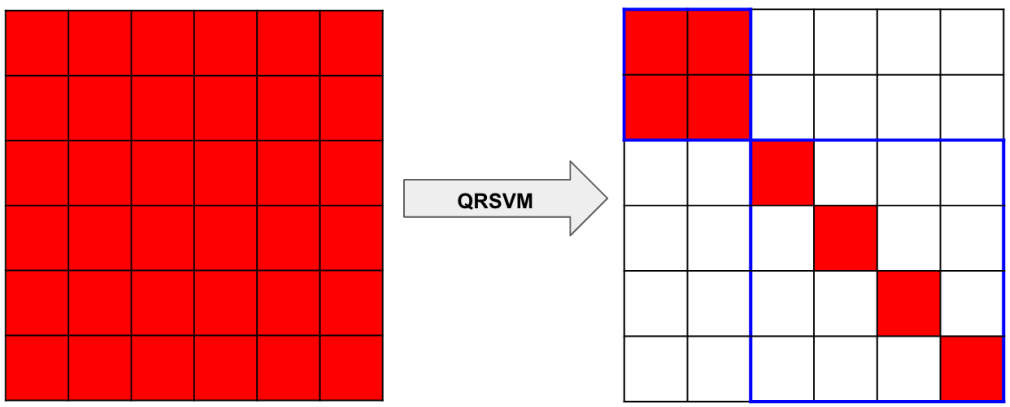
$$\min_{\hat{\alpha}} \frac{1}{2}\hat{\alpha}^T\left(RR^T + \frac{1}{2C}I_n\right)\hat{\alpha} + (\hat{e})^T\hat{\alpha} \tag{3}$$

$$\text{subject to} \quad -Q\hat{\alpha} \leq \mathbf{0}_n$$

Memory-efficient modeling

Structure of Hessian matrix

$$\left(\hat{A}\hat{A}^T + \frac{1}{2C}I_n\right) \Rightarrow \left(RR^T + \frac{1}{2C}I_n\right)$$



Dense $O(n^2)$
Non-separable

Sparse $O(p^2)$
block diagonal separable

Dual Ascent to solve linearly constrained Optimization problem

Lagrangian $\mathcal{L}$ of QRSVM

$$\mathcal{L}(\hat{\alpha}, \beta) = \frac{1}{2}\hat{\alpha}^T\left(RR^T + \frac{1}{2C}I_n\right)\hat{\alpha} + (\hat{e})^T\hat{\alpha} + \beta^T(-Q\hat{\alpha}) \qquad (4)$$

where, $\beta \geq \mathbf{0}_n$ is the Lagrangian dual variable.

Dual Ascent

Dual function: $g(\beta) = \min_{\hat{\alpha}} \mathcal{L}(\hat{\alpha}, \beta)$
Dual Problem: $\max_{\beta} g(\beta)$

Dual Ascent steps

Gradient method - involves iterating through the following steps until convergence (error in β falls below stopping threshold)

Step 1: Minimization of Lagrangian

$$\begin{aligned}\hat{\alpha}^{k+1} &= \arg\min_{\hat{\alpha}} \mathcal{L}(\hat{\alpha}, \beta^k) \\ &= -\left(RR^T + \frac{1}{2C} \times I_n\right)^{-1}(-Q^T\beta^k + \hat{e})\end{aligned} \tag{5}$$

Step 2: Dual variable update

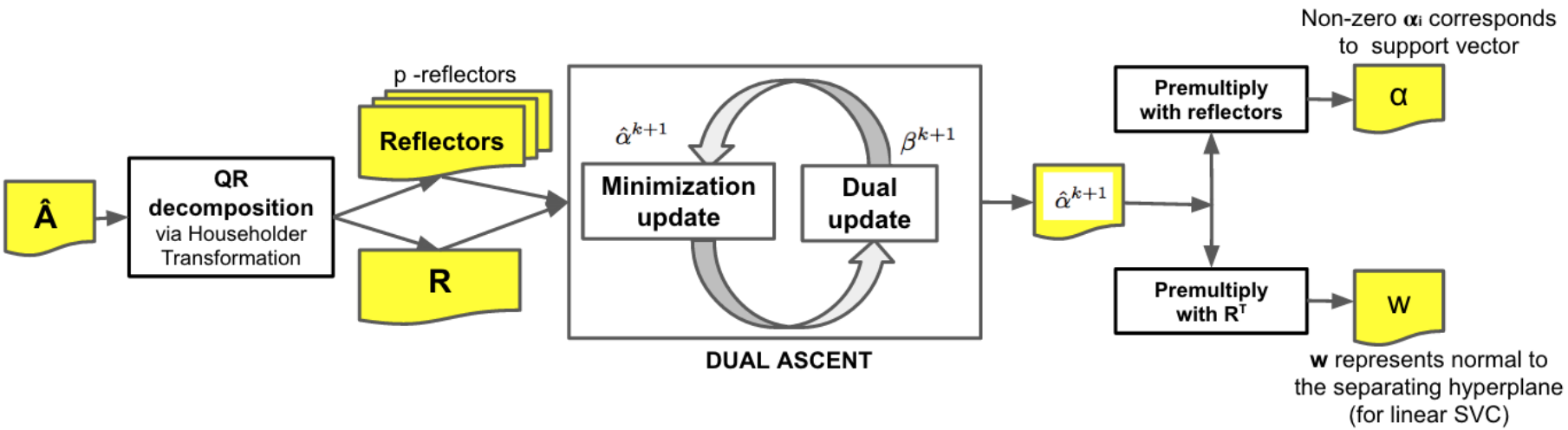
$$\beta^{k+1} = \beta^k + \eta(-Q\hat{\alpha}^{k+1}) \tag{6}$$

$\eta > 0$ is the step size, $\beta^0 = \mathbf{0}_n$.

QRSVM Workflow

Two stages of QRSVM

1. **QR decomposition**: Computational cost $O(np^2)$
2. **Dual Ascent method**: Computational cost $O(np)$/iteration



Optimal Step Size

Fast convergence of Dual Ascent

Based on optimal synchronization period defined for Lazily Synchronous Dual Ascent method , Theorem 1 [Lee, 2016]

Scaling factor for optimal step size

To ensure the minimum number of iterations involving the dual variable update step, the scaling factor $P^\star$ for optimal step size is obtained by

$$P^\star = \max \operatorname*{argmin}_{P \in \mathbb{N}} \max\{|1 - \lambda_{min}(M)P|, |1 - \lambda_{max}(M)P|\} \quad (7)$$

$M := \eta\left(RR^T + \frac{1}{2C} I_n\right)^{-1}$,
$\eta > 0$ is step size
$\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ eigenvalues of matrix M

Optimal step size

For any $\eta > 0$, the optimal step size $\eta^\star$ can be computed using

$$\eta^\star = P^\star \eta, \quad P^\star \in \mathbb{N} \tag{8}$$

where,

$$P^\star = \begin{cases} 1 & \text{if} \quad 0 < \bar{\lambda}^{-1} < 2 \\ \lfloor \bar{\lambda}^{-1} \rfloor & \text{if} \quad \bar{\lambda}^{-1} \geq 2 \end{cases}$$

and $\bar{\lambda} = (\lambda_{max}(M) + \lambda_{min}(M))/2$

$$\bar{\lambda}^{-1} \approx 1/(\eta C)$$

Distributed QRSVM

Stage 1: Distributed QR decomposition

Partition data, $\hat{A}_i \in \mathbb{R}^{\frac{n}{S} \times p}$ on S worker nodes

$$p \ll \frac{n}{S} \implies S \ll \frac{n}{p}$$

Theorem

Given, S horizontal partitions of $\hat{A} = \{\hat{A}_i\}, i = 1..S$,

1. $\hat{A}_i = Q_i R_i$ *at each worker node i*
2. *Gather all R_i's at the Master node*
3. $[R_1; ..; R_S] = Q_g R_g$ *at Master node*

One can represent the factors Q and R of the complete $\hat{A}$ in distributed formulation as

$$Q = \text{diag}(Q_1, Q_2, ..Q_i.., Q_S) \times Q_g$$

$$R = R_g$$

Stage 1: Distributed QR decomposition

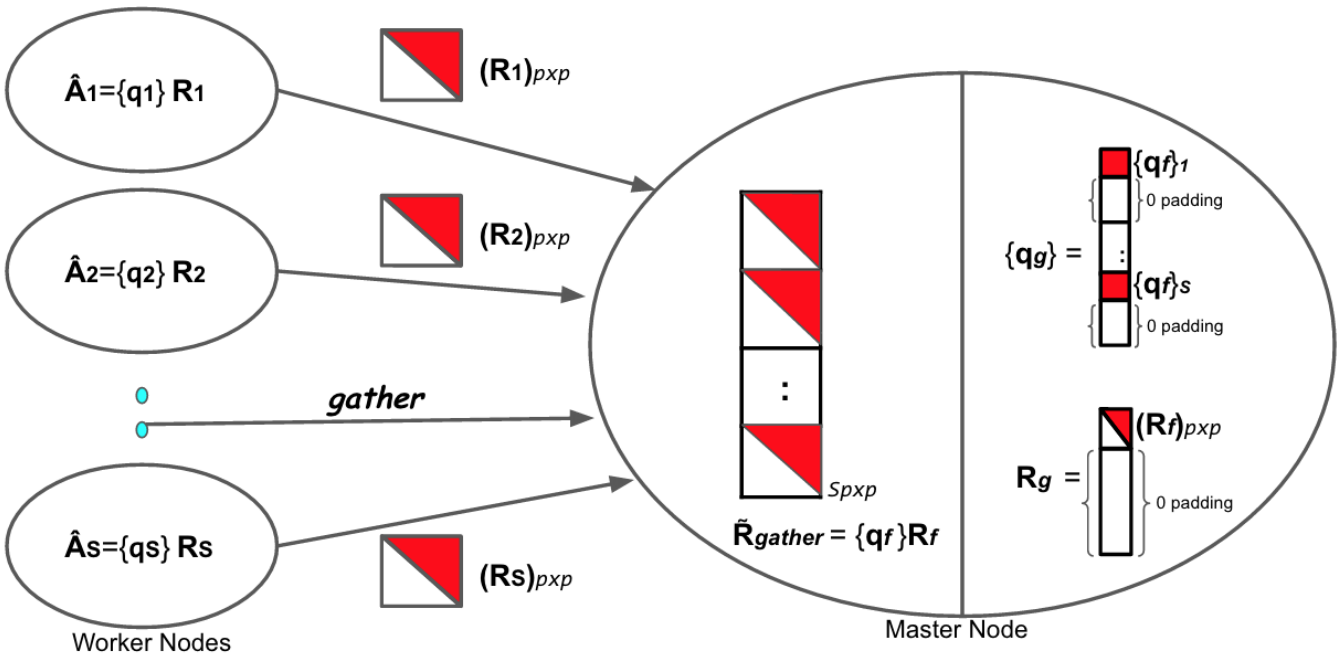


Figure: Implementation

Q_i stored as sets of their Householder reflectors, denoted as $\{q_i\}$

Stage 2: Parallel Dual Ascent

Define, $F = -\left(R_g {R_g}^T + \frac{1}{2C} I_n\right)$

Step 1: Minimization of Lagrangian

Recall,

$$\begin{aligned} \hat{\alpha}^{k+1} &= \arg\min_{\hat{\alpha}} \mathcal{L}(\hat{\alpha}, \beta^k) \\ &= F^{-1}(-Q^T \beta^k + \hat{e}) \end{aligned}$$

Define, $\hat{\beta}^k = Q^T \beta^k$

Stage 2: Parallel Dual Ascent

Define, $F = -\left(R_g {R_g}^T + \frac{1}{2C} I_n\right)$

Partition F into S block-diagonals, $F_i \in \mathbb{R}^{\frac{n}{s} \times \frac{n}{s}}$

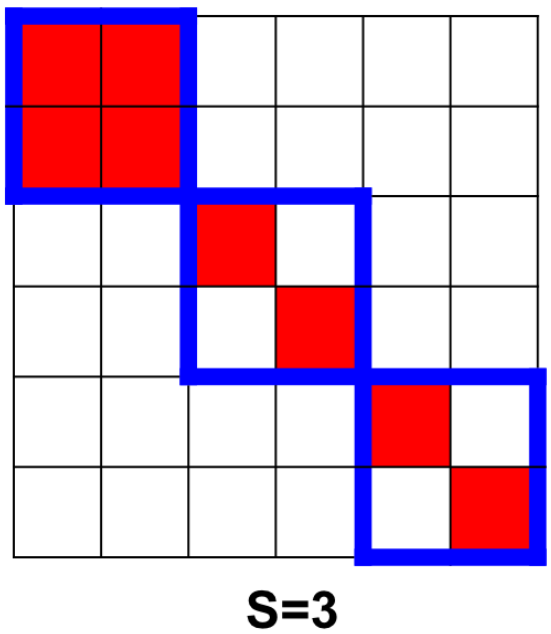


Figure: Block Separable into F_i

Stage 2: Parallel Dual Ascent

Define, $F = -\left(R_g {R_g}^T + \frac{1}{2C} I_n\right)$

Step 1: Minimization of Lagrangian - In Parallel

At compute node, i

$$\hat{\alpha_i}^{k+1} = F_i^{-1}(-\hat{\beta_i}^k + \hat{e_i}) \tag{9}$$

where,

$$F_i{}^{-1} = \begin{cases} F_1{}^{-1} & \text{if} \quad i = 1 \\ -2C & \text{if} \quad i = 2..S \end{cases}$$

Stage 2: Parallel Dual Ascent

Step 2: Dual variable update

Recall,

$$\beta^{k+1} = \beta^k + \eta(-Q\hat{\alpha}^{k+1})$$

Using, $\hat{\beta}^k = Q^T \beta^k$

Stage 2: Parallel Dual Ascent

Step 2: Dual variable update - In Parallel

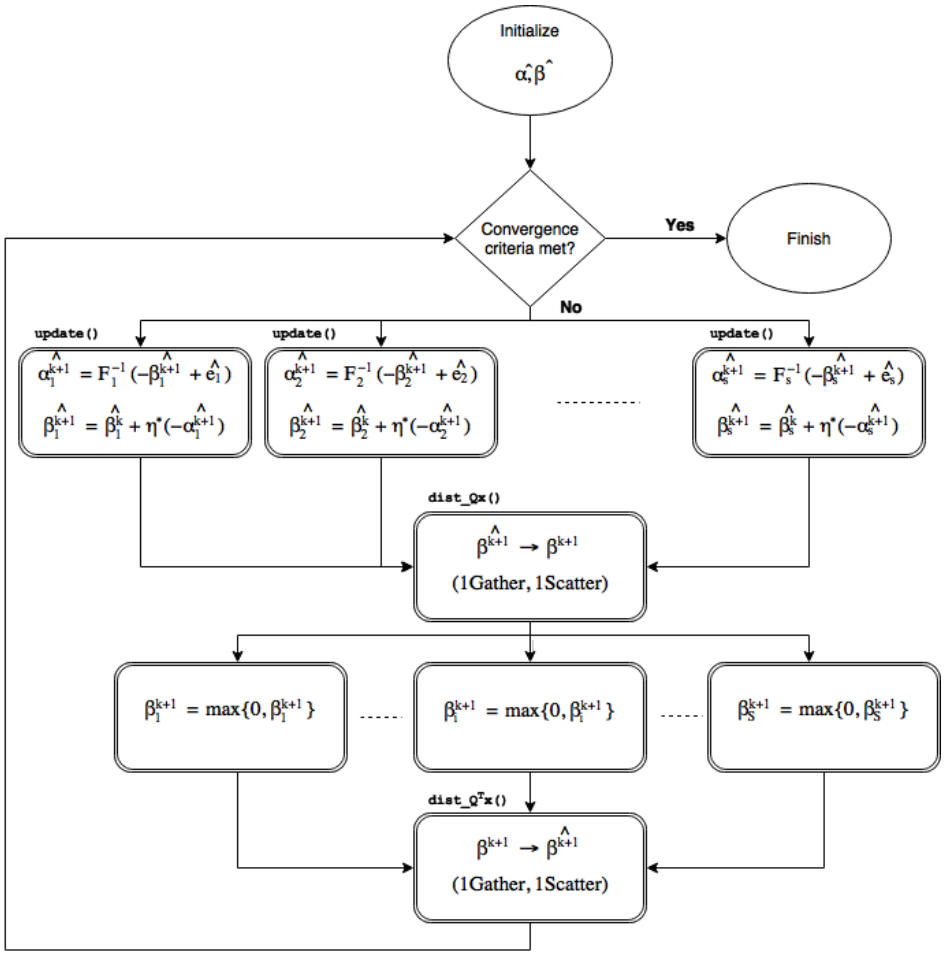
At compute node, i

$$\hat{\beta}_i^{k+1} = \hat{\beta}_i^{k} + \eta^{\star}(-\hat{\alpha}_i^{k+1}) \tag{10}$$

$\eta^{\star}$ is the Optimal step size
$\hat{\beta}^k = Q^T \beta^k$

Stage 2: Parallel Dual Ascent- Implementation

- Local update calculations
- $\hat{\beta}_i$ gather to $\hat{\beta}$
- $\hat{\beta} \Rightarrow \beta$
- β scatter to β_i
- Ensure $\beta_i \geq 0$
- β_i gather to β
- $\beta \Rightarrow \hat{\beta}$
- $\hat{\beta}_i$ scatter to $\hat{\beta}$



Experimental Results

Experimental Setup

Hardware

- *Ada* Supercomputing Cluster at TAMU
- Intel Xeon E5-2670 v2 (Ivy Bridge-EP), 10-core, 2.5GHz
- 64 GB/node and 16 cores/node
- Message-Passing Interface (MPI), InfiniBand interconnect

Dataset	n	d	**Description**
a9a	32560	123	predict annual income
covtype	464810	54	predict forest cover type

Convergence

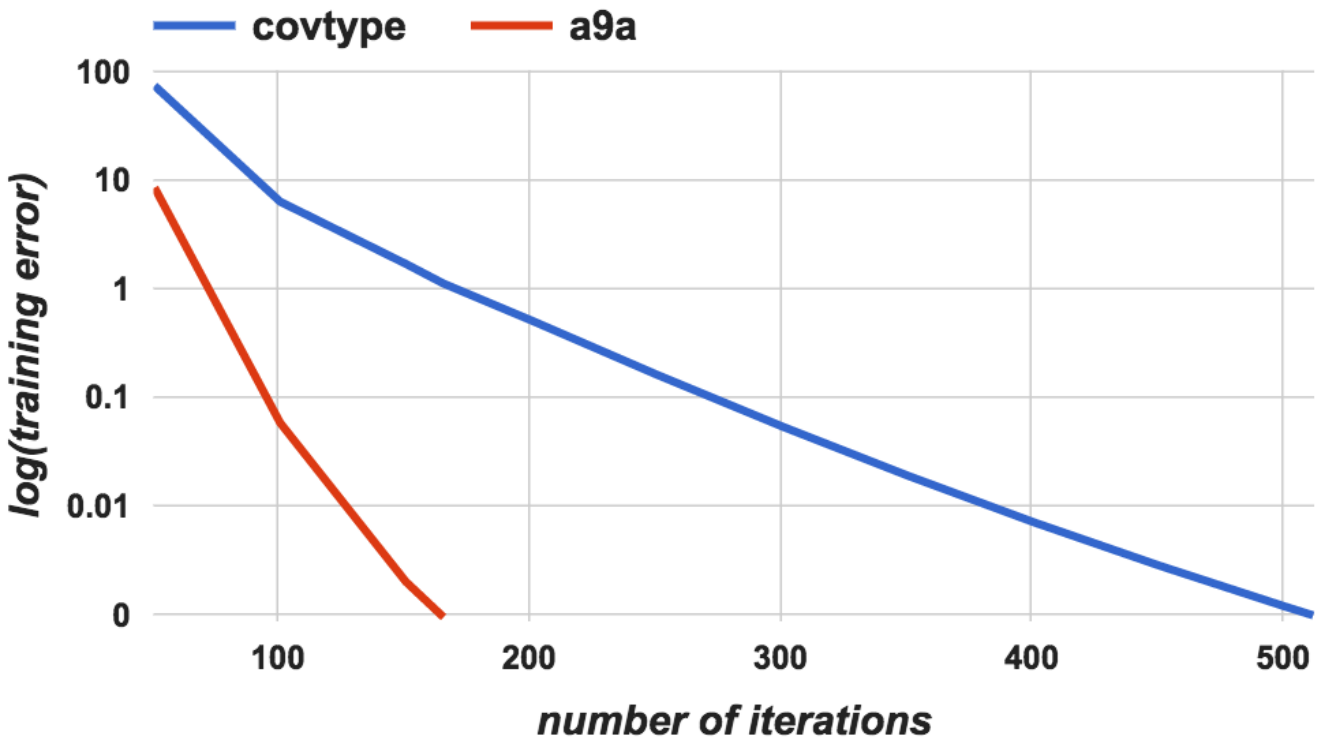


Figure: a9a: k=166 , covtype: k=512, threshold=10^{-3}

Scalability of QRSVM: $O(np^2)$

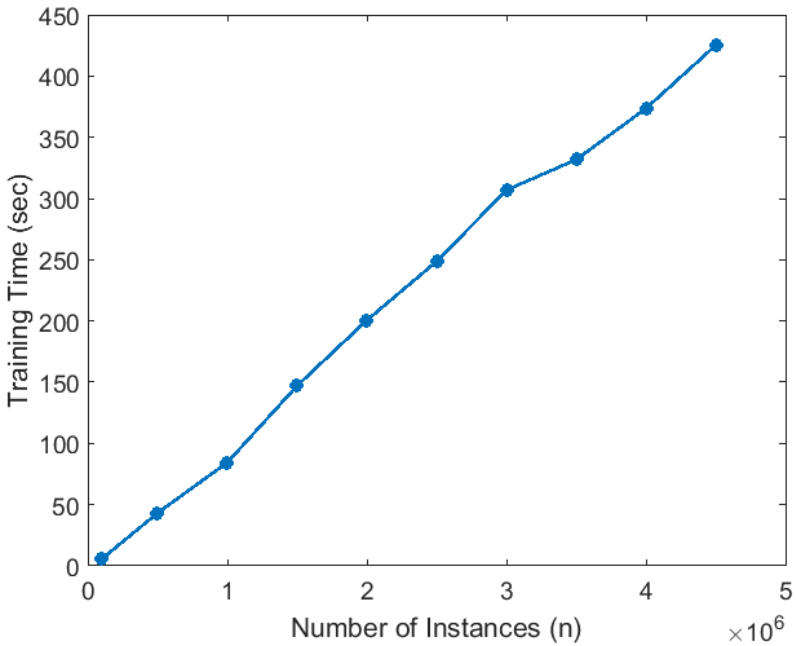


Figure: Scales linearly with n

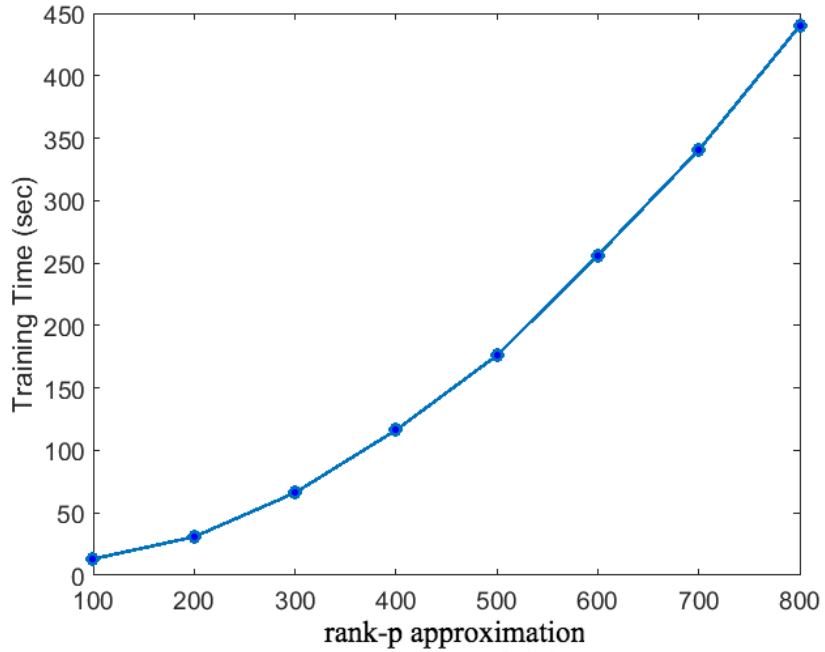


Figure: Scales quadratically with rank p

Optimal Step Size, η^*

Figure: a9a, $\eta^* = 1.9$

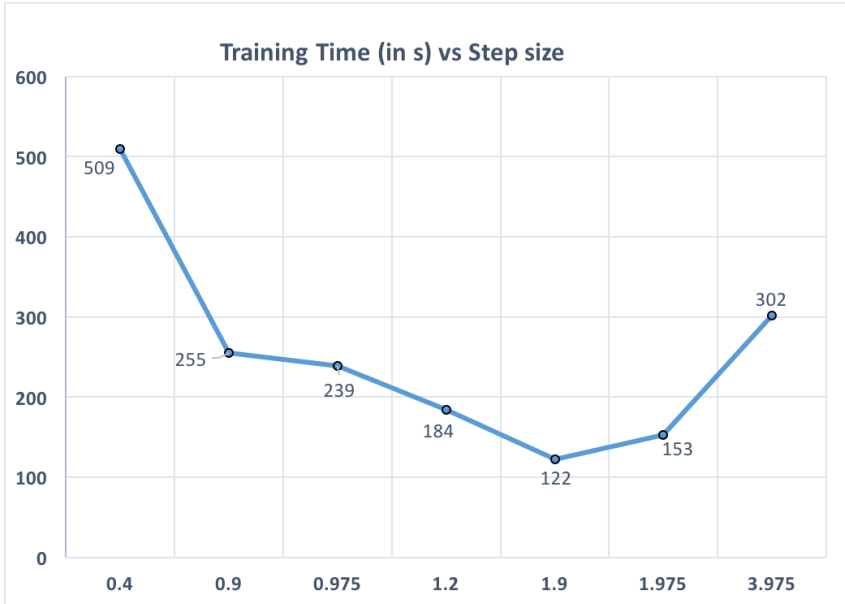


Figure: covType, $\eta^* = 1.9$

Distributed QRSVM: Timing Discussions

Stage 1: Distributed QR

1. Computation: $t(p_{localQR}) + t(p_{masterQR})$
2. Communication: $t(c_{gatherR})$

Stage2: Parallel Dual Ascent

1. Computation: $t(p_{pda})$
2. Communication: $t(c_{pda})$ Gather+Scatter

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Time details	a9a (in ms)	covtype (in s)
$t(p_{meka})$	460	2.1
$t(p_{localQR})$	24	1.89
$t(p_{masterQR})$	4	0.02
$t(c_{gatherR})$	0.5	0.04
$t(p_{pda})$	1628.1	120.18
$t(c_{pda})$	17.1	0.36
$t(train)$	1674.2	122.50

Distributed QRSVM: Parameter Discussions

Parameters	**a9a**	**covtype**
rank, p	40	64
C	2^{-1}	2^{-1}
γ	2^{-3}	2^{3}
approx. K_{error}	0.51	0.58
#processors, S	16	16
stopping threshold	10^{-3}	10^{-3}
optimal step size, η^*	1.9	1.9
#iterations, k	166	512

Comparison with PSVM and P-packSVM ($S = 16$)

Dataset	dis-QRSVM	PSVM	P-packSVM
covType	2 min	20 min	16 min

Demerits of PSVM and P-packSVM

- PSVM uses Incomplete Cholesky Factorization (ICF) $\Rightarrow$ Difficult to parallelize and slow $\Rightarrow$ Unfit for distributed big data analytics
- PSVM training time is $O(n^2)$/iteration $\Rightarrow$ Limited scalability
- P-packSVM solves *primal* form $\Rightarrow$ Slow Convergence

Conclusions

Summary

1. Memory-efficient modeling and training for QRSVM
2. Parallel SVM formulation - distributed QR decomposition and Parallel Dual Ascent
3. Optimal Step size calculation for fast convergence and training
4. Performs significantly better than competing algorithms

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Future Possibilities

1. Can be implemented in **clustered embedded systems/Edge-line devices** to solve large- scale problems rather than using supercomputers
2. QRSVM can be extended for **real-time** data analytics
3. QR decomposition technique can be used for other **Kernel based problems** like Support Vector Regression etc.
4. Motivates for designing **hardware accelerators** to further boost the performance in many domain specific scenarios.

Thank You!

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